

Photons and Fermions in Spacetime with a Compactified Spatial Dimension

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The effects of a nonsimply connected spacetime with the topology of $S^1 \times R^3$ in the vacua of QED and gauged-NJL theories are investigated. It is shown that the polarization effects of twisted and untwisted fermions in QED are equivalent, once the corresponding stable vacuum solution of each fermion class is taken into account. The photon propagation in QED is found to be anisotropic and characterized by several massive photon modes and a superluminal transverse mode. At small compactification radius the masses of the massive modes increase as the inverse of the radius, while the massless photon mode has a superluminal velocity that increases logarithmically with that distance. At low energies the photon masses lead to an effective confinement of the gauge fields into a $(2+1)$ -dimensional manifold transverse to the compactified direction. In the gauged-NJL model, it is shown that for both twisted and untwisted fermions, the smaller the compactification radius, the larger the critical four-fermion coupling needed to generate a fermion-antifermion chiral symmetry breaking condensate.

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I. INTRODUCTION

A significant effort in the recent history of physics bears on the quest to unify all the known fundamental interactions. This old Einstein's dream found a partial realization in the Standard Model, where the electromagnetic, weak and strong nuclear forces were unified in a single one. The fourth fundamental interaction, gravity, has been an obstacle in that path. Quantum physics in the realm of the Riemann space, which is the natural habitat for gravity, gives rise to an anomalous behavior for the quantum processes where the gravitational quantum particle, the graviton, participates. An incomplete unified scenario, where the four fundamental forces cannot be reduced to a single one, limits the possibility to understand the physics of our universe at the earliest times.

An innovative approach in the search for a compatible way to unify gravity with the other forces was attempted in the early works of Kaluza and Klein [1]. There, the possibility of a physical universe with extra dimensions was considered. In their approach, the extra dimensions were compactified in small distances that are hidden to low energy physics, but manifested as gauge fields in the remnant flat four-dimensional spacetime. More recently, the idea of extra compactified dimensions has been extensively used in supergravity, superstrings and brane theories [3]. In a different development [2], extra dimensions have been also applied at lower energy scales in an attempt to understand hierarchical scales existing between the weak and Planck energies. The main new ingredient of this approach is that only gravitons can propagate in the bulk corresponding to the extra dimensions, while the other gauge fields of the Standard Model are constrained to the four-dimensional wall.

Motivated by the importance of the Kaluza-Klein scenario, the study of QFT in nontrivial spacetime has been the focus of attention of many investigators in recent years. It is well known that the global properties of the

spacetime, even if it is locally flat, can give rise to new physics. A seminal discovery in this direction is the so called Casimir effect [4]. In this phenomenon, an attractive force appears between neutral parallel perfectly conducting plates. The materialized attractive force is mediated by the zero-point fluctuations of the electromagnetic field in vacuum. Hence, the Casimir force is interpreted as a macroscopic manifestation of the vacuum structure of the quantized fields in the presence of domains restricted by boundaries or nontrivial topologies [5].

As known, QFT in spacetime with non-trivial topology has nonequivalent types of fields with the same spin [6]. The allowed number of distinct field configurations is determined by the topological structure of the spacetime; generally being more than one in non-simply connected spaces. In particular, for a fermion system in a spacetime which is locally flat but with topology represented by the domain $S^1 \times R^3$ (i.e. a Minkowskian space with one of the spatial dimensions compactified in a circle S^1 of finite length a), the non-trivial topology is transferred into boundary conditions for fermions that are either periodic (untwisted) or antiperiodic (twisted)

$$\psi(t, x, y, z - a/2) = \pm \psi(t, x, y, z + a/2), \quad (1)$$

while for vector fields, only untwisted configurations are allowed. In (1) the compactified dimension with length a has been taken along the \mathcal{OZ} -direction.

Quantum electrodynamics (QED) with photons coupled to untwisted fermions or to a combination of twisted and untwisted fermions is an unstable theory [7]. The instability arises due to polarization effects of untwisted electrons which produce tachyonic electromagnetic modes [7]. For self-interacting scalar fields the space periodicity can also produce instabilities causing a symmetry breaking that makes the massless field to become massive [8]. The acquired mass depends on the periodicity length, so this phenomenon is called topological mass

generation.

To understand in qualitative terms how the fermion boundary conditions in a non-trivial topology can produce instabilities in QED, we should have in mind that, thanks to the vacuum polarization, the photon exists during part of the time as a virtual e^+e^- -pair. The virtual pair can then transfer the properties of the quantum vacuum, which as known, depend on the non-trivial topology and boundary conditions of the space under consideration, to the photon spectrum.

Our main goal in the present report is to analyze the consequences of the non-trivial topology for photon propagation in QED and for fermion condensation in a gauged-NJL theory. We will show that the non-simply connected character of the spacetime may give rise to different photon modes of propagation, which are normally absent in QED in a flat space with trivial topology. Another topic that we will discuss is how the compactified dimension influences the chiral symmetry breaking in a gauged-NJL theory.

II. NON-TRIVIAL VACUUM SOLUTIONS IN COMPACTIFIED QED

The vacuum polarization in the non-trivial spatial topology $S^1 \times R^3$ can be influenced by both virtual untwisted and twisted e^+e^- -pairs. The results for twisted fermions can be easily read off the results at finite temperature, since in the Euclidean space the two theories are basically the same after the interchange of the four-space subindexes $3 \leftrightarrow 4$. Nevertheless, a different situation occurs with untwisted fermions that has no analogy in the statistical case. Henceforth, we concentrate our attention in the untwisted-fermion case.

Let us consider the QED action in a spacetime domain with compactified dimension of length a in the $\mathcal{O}\mathcal{Z}$ -direction

$$\mathcal{S} = \int_{-a/2}^{a/2} dx_3 \int_{-\infty}^{\infty} dx_0 d^2 x_{\perp} \left[-\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\partial - eA - m)\psi \right]. \quad (2)$$

When this compactified QED action is considered for untwisted fermions, the effect of vacuum polarization upon photon propagation gives rise to a tachyonic mass for the third component of the photon field [7]. In a quantum theory the existence of tachyonic modes are an indication that the considered vacuum is not the physical one, and that a symmetry breaking mechanism is in order. Indeed, in compactified QED with untwisted fermions it has been shown [9] that a constant expectation value of the electromagnetic potential component along the compactified direction minimizes the effective potential, thereby stabilizing the theory. The same stable vacuum solution obtained in QED with $S^1 \times R^3$ topology in Ref. [9], is also present in the case of massless QED

with periodic fermions on a circle [10] (QED with $S^1 \times R^4$ topology). Notice that, even though a constant vacuum configuration has $F_{\mu\nu} = 0$, it cannot be gauged to zero, because the gauge transformation that would be needed does not respect the periodicity of the function space in the $S^1 \times R^3$ domain. This is a sort of Aharonov-Bohm effect that makes A_{μ} a dynamical variable due to the non-simply connected topology of the considered spacetime. The lack of gauge equivalence between a constant component of the gauge potential (A_0 in this case) and zero is also manifested in QED at finite temperature and/or density due to the compactification of the time coordinate [11]. In the statistical case, however, the minimum of the potential is at $A_0 = 0$, since only twisted fermions are allowed. On the other hand, in the electroweak theory with a finite density of fermions, a non-trivial constant vacuum A_0 is induced by the fermion density and cannot be gauged away [12]. There, in contrast to the system considered in the present paper, an additional parameter (a leptonic and/or baryonic chemical potential) is needed to trigger the non-trivial constant minimum for A_0 .

To find the physical vacuum that stabilizes the untwisted fermion theory, we propose, following Ref. [9], the following ansatz [13]

$$\bar{A}_{\nu} = \Delta \delta_{\nu 3}, \quad (3)$$

with Δ an arbitrary constant that will be determined from the minimum equation of the effective potential. Due to the periodicity of the A_{μ} fields in the $S^1 \times R^3$ space, the gauge transformations $A_{\mu} \rightarrow A_{\mu} - \frac{1}{e} \partial_{\mu} \alpha$ are restricted to those satisfying $\alpha(x_3 + a) = \alpha(x_3) + 2l\pi$, $l \in \mathbb{Z}$ [11]. Thus, the gauge transformation $\alpha(x) = (x \cdot n)e\Delta$, which connects the constant field configuration (3) with zero, could not satisfy the required periodicity condition unless Δ were given by

$$\Delta = \frac{2l\pi}{ea}, \quad l \in \mathbb{Z} \quad (4)$$

Let us consider then the one-loop effective potential of the theory (2) around the vacuum configuration (3)

$$V = -\frac{1}{2} a^{-1} \ln \left(\text{Det } \bar{G}^{-1} \right). \quad (5)$$

Here $\text{Det } \bar{G}^{-1} = \sum d^4 p \det \bar{G}^{-1} = \sum_{p_3} \int d^3 p \det \bar{G}^{-1}$, with $p_3 = 2n\pi/a$, ($n = 0, \pm 1, \pm 2, \dots$) being the discrete frequencies associated with periodic fermions. In (5) $\bar{G}^{-1} = \gamma \cdot \bar{p} + m$ is the fermion inverse Green's function in the background Δ , with $\bar{p}_{\mu} = (p_0, \mathbf{p}_{\perp}, p_3 - e\Delta)$.

After the Wick rotation to Euclidean space and summing in p_3 we obtain

$$V(\Delta) = -4 \int_{-\infty}^{\infty} \frac{d^3 \hat{p}}{(2\pi)^3} \left[\frac{\varepsilon_p}{2} + a^{-1} \text{Re} \ln \left(1 - e^{-a(\varepsilon_p - ie\Delta)} \right) \right], \quad (6)$$

where $d^3\hat{p} = idp_4d^2p_\perp$ and $\varepsilon_p = \sqrt{\hat{p}^2 + m^2}$.

The extremum of the renormalized effective potential satisfies

$$\begin{aligned} & \frac{\partial V(\Delta)}{\partial \Delta} \Big|_{\Delta=\Delta_0} = \\ &= - \int_{-\infty}^{\infty} \frac{d^3\hat{p}}{(2\pi)^3} \frac{2e^{-a\varepsilon_p} \sin(ae\Delta_0)}{1 + e^{-2a\varepsilon_p} - 2e^{-a\varepsilon_p} \cos(ae\Delta_0)} = 0, \quad (7) \end{aligned}$$

The extrema of Eq. (7) are $\Delta_0 = \frac{l\pi}{ea}$, $l \in \mathbb{Z}$. Nevertheless, the minimum condition $\partial^2 V(\Delta_{\min})/\partial^2 \Delta > 0$ is only fulfilled by the subset

$$\Delta_{\min} = \frac{(2l+1)\pi}{ea}, \quad l \in \mathbb{Z} \quad (8)$$

The elements in the set of minima (8) are gauge equivalent, since they are all connected by allowed gauge transformations ($\alpha(x_3+a) = \alpha(x_3)+2l\pi$). It should be pointed out, however, that the solutions (8) are not gauge equivalent to the trivial vacuum $\Delta = 0$, since none of them satisfies (4). That is, the trivial vacuum belongs to a different gauge class.

Substituting with the minimum solution (8) on Eq. (6) we obtain

$$V(\Delta_{\min}) = -4 \int_{-\infty}^{\infty} \frac{d^3\hat{p}}{(2\pi)^3} \left[\frac{\varepsilon_p}{2} + a^{-1} \text{Re} \ln(1 + e^{-a\varepsilon_p}) \right]. \quad (9)$$

The expression (9) coincides with the one-loop effective potential of the theory (2) for twisted fermions. As expected, in the $am \ll 1$ approximation, the effective potential (9) reduces to

$$V(\Delta_{\min}) = -\frac{7\pi^2}{360a^4}, \quad (10)$$

which is the result reported for twisted fermions in Ref. [7]. Thus, the vacuum energy of both classes of fermions coincides if the corresponding correct vacuum solution is used.

Notice that the parameter Δ only appears in the distribution functions associated to the sums in p_3 . Therefore, when Δ is evaluated at the minimum solution (8), it turns the statistics of the untwisted fermions into that of the twisted ones.

III. PHOTON PROPAGATION IN $S^1 \times R^3$ SPACETIME

To study the photon propagation in the $S^1 \times R^3$ domain, we should solve the dispersion relations of the electromagnetic modes, which in the low-frequency limit has the general form

$$k_0^2 - \mathbf{k}^2 + \Pi(\mathbf{k}^2) = 0, \quad (11)$$

where $\Pi(\mathbf{k}^2)$ accounts for vacuum polarization effects. Different external conditions, as external fields, geometric boundary conditions, temperature, etc., may modify the vacuum and produce, through $\Pi(\mathbf{k}^2)$, a variation in the spectrum of the photon modes.

The solution of the photon dispersion equations (11) can be obtained as the poles of the photon Green's function. Due to the explicit Lorentz symmetry breaking in the $S^1 \times R^3$ topology, we must consider, in addition to the usual tensor structures k_μ and $g_{\mu\nu}$, a space-like unit vector pointing along the compactified direction $n^\mu = (0, 0, 0, 1)$. Then, the general structure of the electromagnetic field Green's function is

$$\begin{aligned} \Delta_{\mu\nu}(k) &= P(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) + \\ &+ Q[\frac{k_\mu k_\nu}{k^2} - \frac{k_\mu n_\nu + n_\mu k_\nu}{(k \cdot n)} + \frac{k^2 n_\mu n_\nu}{(k \cdot n)^2}] + \frac{\alpha}{k^4} k_\mu k_\nu, \quad (12) \end{aligned}$$

where α is a gauge fixing parameter corresponding to the covariant gauge condition $\frac{1}{\alpha} \partial_\mu A_\mu = 0$, and P and Q are defined as

$$\begin{aligned} P &= \frac{1}{k^2 + \Pi_0}, \\ Q &= -\frac{\Pi_1}{(k^2 + \Pi_0) \{k^2 + \Pi_0 - \Pi_1[k^2/(k \cdot n)^2 + 1]\}}. \quad (13) \end{aligned}$$

The parameters Π_0 and Π_1 are the coefficients of the polarization operator $\Pi_{\mu\nu}$, which in the $S^1 \times R^3$ space can be written as

$$\begin{aligned} \Pi_{\mu\nu}(k) &= \Pi_0(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) + \\ &+ \Pi_1[\frac{k_\mu k_\nu}{k^2} - \frac{k_\mu n_\nu + n_\mu k_\nu}{(k \cdot n)} + \frac{k^2 n_\mu n_\nu}{(k \cdot n)^2}]. \quad (14) \end{aligned}$$

From (12)-(13) the photon dispersion relations are

$$k_0^2 - \mathbf{k}^2 + \Pi_0 = 0, \quad (15)$$

$$k_0^2 - \mathbf{k}^2 + (\Pi_0 - \frac{\widehat{k}^2}{k_3^2} \Pi_1) = 0, \quad (16)$$

with $\widehat{k}^2 = k_0^2 - k_\perp^2$ and $k_\perp^2 = k_1^2 + k_2^2$. We point out that in addition to the transverse mode associated to Eq. (15) (normally present in Minkowski spacetime with trivial topology), a longitudinal mode, Eq. (16), arises here due to the presence of the extra coefficient Π_1 . The situation resembles the finite temperature case. Nevertheless, as

discussed below, the physical consequences of the spatial compactification are radically different from those at finite temperature.

The compactified \mathcal{OZ} -direction distinguishes itself from the other spatial directions, so it is convenient to separate the analysis between photons propagating along \mathcal{OZ} ($k_\perp = 0$), and photons propagating perpendicularly to that direction ($k_3 = 0$).

The dispersion relations (15) and (16) for photons propagating perpendicularly to the compactified direction ($k_3 = 0$) are found, from Eqs.(14)-(16), to reduce respectively to

$$k_0^2 - k_\perp^2 - \frac{\hat{k}^2}{k_\perp^2} \Pi_{00} = 0, \quad (17)$$

$$k_0^2 - k_\perp^2 - \Pi_{33} = 0. \quad (18)$$

To find the solutions of Eqs. (17)-(18) at the one-loop level, we need to calculate the one-loop polarization operator components Π_{00} and Π_{33} for untwisted fermions. Considering the free propagator of untwisted fermions at the minimum solution (8)

$$\tilde{G}(x - x') = \frac{1}{(2\pi)^3 a} \sum d^4 p \exp[i p \cdot (x - x')] G(\tilde{p}), \quad (19)$$

where

$$G(\tilde{p}) = \frac{\tilde{p} - m}{\tilde{p}^2 - m^2 + i\epsilon}, \quad \tilde{p}_\mu = (p_0, p_\perp, p_3 - e\Delta_{\min}), \quad (20)$$

and $\sum p^4 = \sum_{p_3} d^3 p$, $p_3 = 2n\pi/a$, ($n = 0, \pm 1, \pm 2, \dots$) being the discrete frequencies associated to periodic fermions, the corresponding one-loop polarization operator is given by

$$\begin{aligned} \Pi_{\mu\nu}(k) &= \\ &= -\frac{4ie^2}{(2\pi)^3 a} \sum d^4 p \left\{ \frac{\tilde{p}_\mu (\tilde{p}_\nu - k_\nu) - \frac{1}{2} [\tilde{p} \cdot (\tilde{p} - k) - m^2] g_{\mu\nu}}{(\tilde{p}^2 - m^2) [(\tilde{p} - k)^2 - m^2]} \right. \\ &\quad \left. + \mu \leftrightarrow \nu \right\} \end{aligned} \quad (21)$$

In the $a |\hat{k}| \ll am \ll 1$ limit, we obtain

$$\begin{aligned} \Pi_{00}(k_3 = 0, k_0 = 0, k_\perp \sim 0) &\simeq \frac{e^2}{3\pi^2} k_\perp^2 \left[\frac{1}{2} \ln \xi + \mathcal{O}(\xi^0) \right] + \\ &\quad + \mathcal{O}(k_\perp^4) \end{aligned} \quad (22)$$

$$\Pi_{33}(k_3 = 0, k_0 = 0, k_\perp \sim 0) \simeq \frac{e^2}{a^2} \left[\frac{1}{3} + \mathcal{O}(\xi^2) \right] + \mathcal{O}(k_\perp^2) \quad (23)$$

where $\xi = am/2\pi \ll 1$. Using the results (22) and (23) in the dispersion equations (17), (18), and taking into account that the photon velocity for each propagation mode can be obtained from $v(\mathbf{k}) = \partial k_0 / \partial |\mathbf{k}|$, we find that within the considered approximation the transverse and longitudinal modes propagate perpendicularly to the compactified direction with velocities

$$v_\perp^T \simeq 1 - \frac{e^2}{12\pi^2} \ln \xi, \quad (24)$$

$$v_\perp^L \simeq 1 - [(M_\perp^L)^2 / 2k_\perp^2], \quad (25)$$

respectively, where $(M_\perp^L)^2 = \Pi_{33} = e^2/3a^2 > 0$ plays the role of an effective topological mass for the longitudinal mode.

We call the reader's attention to the fact that the modifications found for the two velocities, v_\perp^T and v_\perp^L , have different origins. The modification of the longitudinal velocity v_\perp^L is due to the appearance of the topological mass M_\perp^L ; while the transverse superluminal velocity v_\perp^T (note that $v_\perp^T > c$ because $\xi < 1$ in the used approximation) appears as a consequence of a genuine variation of the refraction index in the considered spacetime. Modifications of the photon speed in non-trivial vacua have been previously reported in the literature [15].

It is easy to corroborate that for compactification lengths in agreement with the used approximation, $a < 1/m \sim 10^3 \text{ fm}$, the transverse velocity (24) is about 0.1% larger than the light velocity in trivial spacetime. We underline that albeit $v_\perp^T > c$, there is no causality violation in this problem. To understand this, let us recall that the velocity (24) is a low-frequency mode velocity. On the other hand, the velocity of interest for signal propagation, and hence the relevant one for causality, is the high-frequency velocity $v_\perp^T(q_0 \rightarrow \infty)$. To determine the difference between the two, one would need to investigate the absorption coefficient, $\text{Im}[n(q_0)]$, with $n(q_0)$ being the refraction index as a function of the frequency in the space with $S^1 \times R^3$ topology. However, aside from any needed calculation, we agree with the analysis of Refs. [16] regarding the lack of causality violations in similar systems. We believe that in the case under study no (micro-)causality should be violated, because the events taking place in the $S^1 \times R^3$ space are not constrained by the null cone of a Minkowskian system, as Lorentz symmetry is explicitly broken in the present situation.

The low-frequency limit ($k_0 = 0, |\mathbf{k}| \rightarrow 0$) used to obtain the longitudinal-mode mass M_\perp^L is essential to study the static properties of the electromagnetic field in this space. The mass obtained in this limit plays the role of a magnetic mass of the longitudinal electromagnetic mode [17]. As showed in Ref. [9], this topological mass affects the magnetic response of the system.

Considering now photons propagating along the \mathcal{OZ} -direction ($k_\perp = 0$), the dispersion relations (15) and (16) can be written respectively as

$$k_0^2 - k_3^2 - \Pi_{11} = 0 \quad (26)$$

$$k_0^2 - k_3^2 - \frac{k_0^2 - k_3^2}{k_0^2} \Pi_{33} = 0 \quad (27)$$

Assuming $ak_0 \ll am \ll 1$ in (21), the components of the polarization operator appearing in (26) and (27) become

$$\Pi_{11}(k_\perp = 0, k_3, k_0 \sim 0) \simeq \frac{e^2}{a^2} \left[\frac{1}{9} + \mathcal{O}(\xi^2) \right] + \mathcal{O}(k_0^2) \quad (28)$$

$$\Pi_{33}(k_\perp = 0, k_3, k_0 \sim 0) \simeq \frac{e^2}{a^2} \frac{k_0^2}{k_3^2} \left[\frac{-1}{9} + \mathcal{O}(\xi^2) \right] + \mathcal{O}(k_0^4) \quad (29)$$

From (26), (27), (28) and (29) we can straightforwardly find the low-frequency limit of the photon velocities for transverse and longitudinal modes propagating along the $\mathcal{O}\mathcal{Z}$ -direction

$$v_\parallel^T \simeq 1 - \left[(M_\parallel^T)^2 / 2k_3^2 \right], \quad (30)$$

$$v_\parallel^L \simeq 1 - \left[(M_\parallel^L)^2 / 2k_3^2 \right], \quad (31)$$

where both transverse and longitudinal mode masses coincide and are given by $M_\parallel^T = M_\parallel^L = e/3a$. We stress that in this case both velocities are smaller than the light velocity in trivial Minkowski space c , and that the modification is due, as in (25), to the appearance of a topological photon mass for each mode.

IV. EFFECT OF COMPACTIFICATION ON FERMION CONDENSATION

Dynamical chiral symmetry breaking in phenomenological models with fermion interactions of Nambu-Jona-Lasinio (NJL) type have attracted a great deal of attention [18], [19] since its introduction in the seminal paper of Nambu and Jona-Lasinio [20]. Our interest here is to consider dynamical chiral symmetry breaking in a non-trivial topological space taking into account the results of Sec. II. With this aim, let us add a NJL four-fermion term with coupling G to massless QED, so that its Lagrangian density becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\partial - eA)\psi + \frac{G}{2N} \left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\psi)^2 \right]. \quad (32)$$

The fermions in (32) are assumed to carry out a flavor index $\alpha = 1, 2, \dots, N$. Introducing the composite fields

$\sigma = -\frac{G}{N} (\bar{\psi}\psi)$, $\pi = -\frac{G}{N} (\bar{\psi}i\gamma^5\psi)$, the gauged-NJL Lagrangian density (32) can be rewritten as

$$\mathcal{L}_{NJL} = \frac{1}{2} [\bar{\psi}, i\gamma^\mu D_\mu \psi] - \bar{\psi} (\sigma + i\gamma^5 \pi) \psi - \frac{N}{2G} (\sigma^2 + \pi^2). \quad (33)$$

The Lagrangian (32) has a continuous chiral symmetry $\psi \rightarrow e^{i\alpha\gamma^5} \psi$, but it is clear from Eq. (33) that if σ gets a different from zero vacuum expectation value (vev) $\bar{\sigma}$, this chiral symmetry is broken and the fermions acquire mass. We are interested in the effective potential in the large- N limit, assuming $e \ll G$. The effective potential is a function of the scalar and pseudo-scalar fields σ and π respectively, and can be obtained by integrating out all the fluctuation fields in the path integral. In a flat and topologically trivial spacetime, the effective potential in leading order at large- N , is given, after performing the Wick rotation to Euclidean space, by

$$V(\sigma) = \frac{\sigma^2}{2G} - 2 \int^\Lambda \frac{d^4 p_E}{(2\pi)^4} \ln(\sigma^2 + p_E^2), \quad (34)$$

where Λ is a large momentum cutoff. In expression (34) we dropped all the σ -independent terms, as they will not contribute to the stationary solution of the potential. We considered a configuration with $\pi = 0$ and $\sigma = \text{constant}$, since the effective potential V only depends on the chiral invariant $\rho^2 = \sigma^2 + \pi^2$.

The vacuum solution is determined by the stationary point of the effective potential (34). At $G > G_c = 4\pi^2/\Lambda^2$, the stationary equation $\partial V(\sigma)/\partial\sigma = 0$ has a non-trivial solution $\bar{\sigma}$ that corresponds to a global minimum of (34).

If the third spatial dimension is compactified in a circle of radius a , the potential (34) for antiperiodic fermions becomes

$$V_{AP}(\sigma) = \frac{\sigma^2}{2G} - 2 \int^\Lambda \frac{d^4 p_E}{(2\pi)^4} \ln(\sigma^2 + p_E^2) - \frac{4}{a} \int \frac{d^3 \hat{p}}{(2\pi)^3} \ln(1 + e^{-a\bar{\varepsilon}_p}), \quad (35)$$

with $\bar{\varepsilon}_p = \sqrt{\hat{p}^2 + \sigma^2}$. The last term in the RHS of Eq. (35) is obtained after summing in the discrete momentum p_3 . The appearance of this new term gives rise to the critical coupling $G_c^a = 6a^2 G_c / (6a^2 - G_c)$, which now depends on the compactification radius a . Notice that $G_c^a > G_c$. Hence, for twisted fermions the compactification tends to restore the symmetry, in agreement with results previously found in Refs. [19], [21] within a pure NJL theory (without gauge fields). When periodic fermions are considered in the trivial vacuum, the corresponding effective potential V_P , is similar to (35), with the only change of a negative sign in front of the exponential in the RHS of Eq. (35). This case was also studied in Ref. [19] in the context of a pure NJL model. There, the analysis of the minimum of the potential revealed that the effect

of the compactified dimension is to enhance the chiral condensation, i.e., to decrease the critical value of the coupling.

The situation is different however for the gauged-NJL theory with untwisted fermions. Here, in analogy with QED, the stable vacuum is given by the constant vector potential (8). Consequently, the effective potential must explicitly depend on a nontrivial vacuum solution. Summing in the discrete momentum p_3 and taking into account that the constant vacuum (3) enters in the calculation of the effective potential as a shift $p_3 \rightarrow p_3 + \Delta$ in this discrete variable, the effective potential results

$$V_P(\sigma, \Delta) = \frac{\sigma^2}{2G} - 2 \int^\Lambda \frac{d^4 p_E}{(2\pi)^4} \ln(\sigma^2 + p_E^2) - \frac{4}{a} \int \frac{d^3 \hat{p}}{(2\pi)^3} \ln [1 - e^{-a(\bar{\varepsilon}_p - ie\Delta)}]. \quad (36)$$

Notice that when the minimum solution (8) is substituted on (36), one obtains $V_P(\sigma, \Delta_{\min}) = V_{AP}(\sigma)$. As a consequence, the critical coupling for the chiral condensation with untwisted fermions reduces to the same G_c^a already found for twisted fermions in the trivial vacuum. Thus, we conclude that, independently of the fermion boundary condition, the effect of the compactification in the theory (32) is to counteract the condensation, and eventually, to reinstate the chiral symmetry at some critical value of the compactification radius.

V. CONCLUDING REMARKS

In this paper we have shown that in a nonsimply connected spacetime with topology $S^1 \times R^3$, the stable vacuum solution for QED with untwisted fermions is given by constant field configurations that are gauge equivalent to $\bar{A}_3 = \frac{\pi}{ea}$, while for twisted fermions the stable solutions correspond to constant gauge configurations equivalent to the trivial vacuum. As a consequence, the one-loop effective potentials for twisted and untwisted fermions coincide when the corresponding stable vacuum

solutions are considered. A direct implication of the relation between the fermion boundary conditions and the QED vacua is that the vacuum polarization cannot distinguish between the two classes of fermions, once the corresponding true vacuum is taking into account.

Another interesting outcome of this investigation is the anisotropy in the photon propagation due to the non-trivial topology of the spacetime. In the $S^1 \times R^3$ domain, the photons have several massive modes and a transverse superluminal one. The masses of the photon modes increase as the inverse of the compactification radius, while the superluminal velocity of the massless mode increases logarithmically with that topological distance. The existence of massive modes implies that at very small radius of compactification, the photon propagation at low energies is effectively confined to a Minkowskian $(2+1)$ -dimensional manifold, on which only superluminal photons propagate. Therefore, photons moving in such a lower dimensional space experience the lack of Lorentz symmetry of the general manifold ($S^1 \times R^3$) on which the lower-dimensional space is embedded, allowing them to have a group velocity larger than the usual Minkowskian velocity c .

We also considered how the non-trivial topology affects the condensation of fermion-antifermion pairs. This was done in the framework of QED with an additional four-fermion interaction (gauged-NJL theory). In this model we found that the smaller the compactification radius, the larger the critical four-fermion coupling needed to generate a fermion-antifermion chiral symmetry breaking condensate. Contrary to what occurs in a pure NJL model [19], this result is obtained for both twisted and untwisted fermions, once the corresponding stable vacuum is considered. Thus, we conclude that in the gauged-NJL theory the compactification tends to reinstate the chiral symmetry.

The results we are reporting here can be of interest for condensed matter quasi-planar systems, as well as for theories with extra dimensions.

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